MATH 2050C Lecture 7 (Feb 7)

Done: $\mathbb{R}$ is the complete ordered field.

GOAL: Define limit of a sequence $\lim(x_n) = x$

and study limit properties

Def$^{\underline{n}}$: A sequence of real numbers is a function

$$X : \mathbb{N} \to \mathbb{R}$$

Denote: $X(1) := x_1$, $X(2) =: x_2, \ldots,$ $X(n) =: x_n$

Write: $X = (x_n) = (x_1, x_2, x_3, \cdots)$

CAUTION: Sets $\neq$ Sequences

E.g.) $\left((-1)^n\right) = (-1, 1, -1, 1, \ldots)$ ordered & infinite

$\{(-1)^n : n \in \mathbb{N}\} = \{-1, 1\}$ unordered & maybe finite

Examples of sequences

(1) constant seq. $(1, 1, 1, 1, 1, \ldots)$

(2) geometric seq. $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right) = \left(\frac{1}{2^n}\right)$

(3) arithmetic seq. $(1, 5, 9, 13, 17, \dots) = (4n-3)$

→ +ve odd seq. $(1, 3, 5, 7, 9, \dots)$

+ve even seq. $(2, 4, 6, 8, 10, \dots)$

(4) Fibonacci seq. ("inductively defined")

$$x_1 := 1 \quad ; \quad x_2 := 1$$

$$x_n := x_{n-1} + x_{n-2} \quad \text{for } n \geq 3$$

$$(x_n) = (1, 1, 2, 3, 5, 8, 13, 21, \dots)$$

Q: How to define the limit of a sequence?

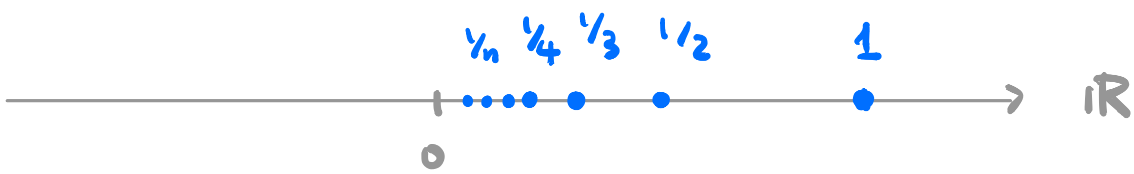
Simplest example:

$$(x_n) = \left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{100}, \dots\right)$$

We want to say: " $\lim\left(\frac{1}{n}\right) = 0$ "

because x_n "eventually" are getting "close" to 0

quantify this!



Def$^{\underline{n}}$: (ε - K definition for limit)

We say that (x_n) converges to $x \in \mathbb{R}$ (finite number)

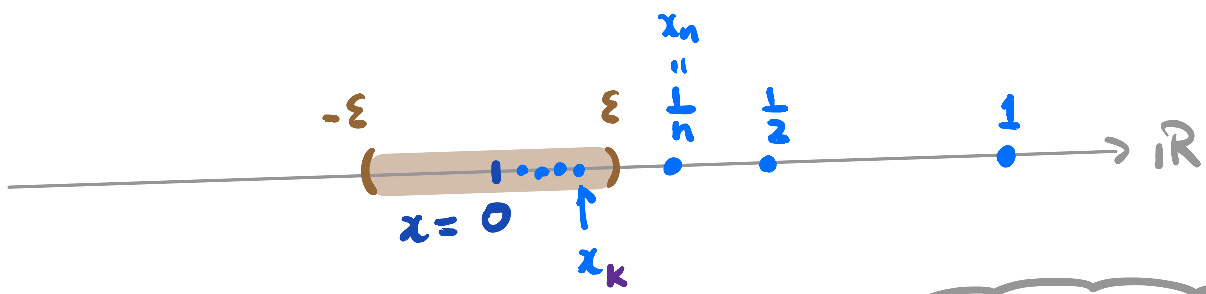
Notation: $\lim (x_n) = x$ or $\lim\limits_{n \to \infty} x_n = x$ or $x_n \to x$

iff $\forall \underset{\text{(small)}}{\varepsilon > 0}$, $\exists \underset{\text{(large)}}{K} = K(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon \qquad \forall n \geq K$$

(i.e. $x - \varepsilon < x_n < x + \varepsilon$)

Picture:



Example: $\boxed{\lim (\frac{1}{n}) = 0}$

$|\frac{1}{n} - 0| < \varepsilon$

$\Leftrightarrow \quad \frac{1}{n} < \varepsilon$

$\Leftrightarrow \quad \frac{1}{\varepsilon} < n$

Let $\varepsilon > 0$ be fixed but arbitrary.

By Archimedean Property, we can choose

$K \in \mathbb{N}$ s.t. $K > \frac{1}{\varepsilon}$. Then, $\forall n \geq K$,

$$|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{K} < \varepsilon$$

$\square$

Example: $\boxed{\lim\left(\frac{3n+2}{n+1}\right) = 3}$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K \overset{*}{>} \frac{1}{\varepsilon}$ (by Archimedean Property)

Then, $\forall\, n \geq K$,

$$\left| \frac{3n+2}{n+1} - 3 \right| = \left| \frac{(3n+2) - 3(n+1)}{n+1} \right| = \left| \frac{-1}{n+1} \right|$$

$$= \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{K} \overset{*}{<} \varepsilon$$

$\square$

Def: Given a seq (x_n) of real numbers, we say

(i) (x_n) is convergent if $\exists\, x \in \mathbb{R}$ s.t. $\lim (x_n) = x$

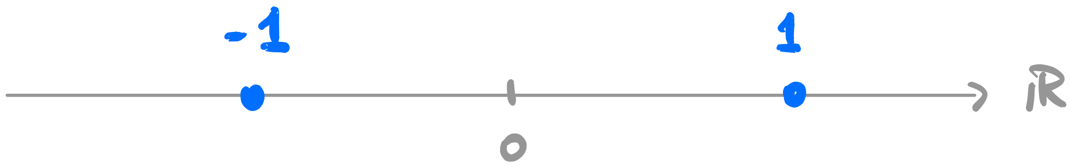
(ii) (x_n) is divergent if (x_n) is NOT convergent

i.e. $\nexists\, x \in \mathbb{R}$ s.t. $\lim (x_n) = x$

Example of divergent seq

Example: $\left((-1)^n\right)$ is divergent.

Picture:



Why 1 is not the limit?

If $\lim((-1)^n) = 1$, then

$\left[\forall \varepsilon > 0, \exists K \in \mathbb{N} \text{ st } |(-1)^n - 1| < \varepsilon \ \forall n \geq K\right] \cdots (*)$

But for n odd, $|(-1)^n - 1| = |(-1) - 1| = 2$

So $(*)$ cannot be true.

Take $\varepsilon = 1 > 0$, if $(*)$ is true, $\exists K = K(1) \in \mathbb{N}$.

but if we choose $n \geq K$ odd, then

$|(-1)^n - 1| = 2 \not< 1$ Contradiction!

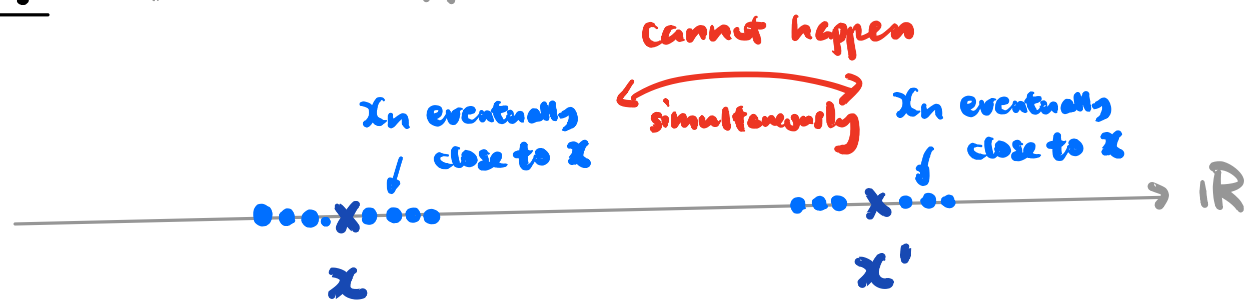
Similarly, -1 is **NOT** the limit either (Pf: Exercise)

Actually, $\nexists x \in \mathbb{R}$ st $\lim((-1)^n) = x$.

(Pf later).

Prop: Any convergent seq. (x_n) has a unique limit.

Proof: Idea: Suppose there are two limits x, x'



Suppose NOT, then $\exists\ x \neq x' \in \mathbb{R}$ s.t.

$$\lim(x_n) = x \quad \& \quad \lim(x_n) = x'$$

Consider $\varepsilon := \dfrac{|x - x'|}{4} > 0$. by def^{n} of limit,

- $\lim(x_n) = x \Rightarrow \exists\ K = K(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon \qquad \forall n \geq K$$

for the same ε

- $\lim(x_n) = x' \Rightarrow \exists\ K' = K'(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x'| < \varepsilon \qquad \forall n \geq K'$$

Take $\bar{K} := \max\{K, K'\} \in \mathbb{N}$. Then, $\forall n \geq \bar{K}$,

$$|x - x'| = |(x_n - x) - (x_n - x')|$$

Contradiction!

$$\overset{(\triangle\text{-ineq})}{\leq} |x_n - x| + |x_n - x'| < \varepsilon + \varepsilon = \frac{|x - x'|}{2}$$

$\square$

Now, we come back to

<u>Claim:</u> $((-1)^n)$ is divergent.

<u>Pf:</u> Suppose NOT, i.e. $((-1)^n)$ is convergent.

By Prop. $\exists$ unique limit $x = \lim((-1)^n)$.

By def$^{\underline{n}}$ of limit, $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t.

$$|(-1)^n - x| < \varepsilon \qquad \forall n \geq K.$$

Fix $\varepsilon = 1 > 0$, then $\exists K \in \mathbb{N}$ s.t.

$$|(-1)^n - x| < 1 \qquad \forall n \geq K.$$

Fix $n_1, n_2 \geq K$ s.t. n_1 odd, n_2 even. Then

$$\begin{aligned} 2 = |-1-1| &= |(-1)^{n_1} - (-1)^{n_2}| \\ &= |((-1)^{n_1} - x) - ((-1)^{n_2} - x)| \\ &\leq |(-1)^{n_1} - x| + |(-1)^{n_2} - x| \\ &< 1 + 1 = 2 \end{aligned}$$

Contradiction! $\blacksquare$

More examples

Example: Let $a > 0$. Then $\boxed{\lim \left(\frac{1}{1+na} \right) = 0}$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K > \frac{1}{a\varepsilon} > 0$

Then, $\forall n \geq K$.

$$\left| \frac{1}{1+na} - 0 \right| = \left| \frac{1}{1+na} \right|$$

$$= \frac{1}{1+na} < \frac{1}{na} \leq \frac{1}{Ka} < \varepsilon$$

$$\left| \frac{1}{1+na} - 0 \right| < \varepsilon$$

$$\frac{1}{1+na} < \frac{1}{na} \leq \frac{1}{Ka} < \varepsilon$$

$$K > \frac{1}{a\varepsilon}$$

$\square$

Example: Let $b \in (0,1)$. Then $\boxed{\lim (b^n) = 0}$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K > \frac{\log \varepsilon}{\log b}$

Then, $\forall n \geq K$.

$$|b^n - 0| = b^n \leq b^K < \varepsilon$$

Want: $|b^n - 0| < \varepsilon$

$$|b^n - 0| = b^n \leq b^K < \varepsilon$$

Solve for K

$$K \log b < \log \varepsilon$$

$$K > \frac{\log \varepsilon}{\log b}$$

$\square$